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SHORT COMMUNICATIONS

Version of the Elimination Method for a System of Partial Differential Equations

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Abstract— For a system of three first-order partial differential equations with three independent variables, we obtain sufficient conditions for one component of the solution to satisfy a third-order Bianchi equation. We also obtain conditions for the solvability of this system by quadratures.

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There is a well-known method for reducing systems of algebraic and ordinary differential equations to a single equation, which is usually called the elimination method. In the present paper, a similar method is applied to the system

$$u_x = av + bw, \qquad v_y = cu + dw, \qquad w_z = eu + fv \tag{1}$$

in the parallelepiped $T = \{(x, y, z) : \ x_0 < x < x_1,\ y_0 < y < y_1,\ z_0 < z < z_1\}$ of three-dimensional Euclidean space, where a, b, c, d, e, and f are given continuous functions of the variables (x, y, z) in $\overline{T}$. (The smoothness classes of these functions will be indicated below.) System (1) can be viewed as an analog of the system

$$u_x = \alpha u + \beta v + f_1, \qquad v_y = \gamma u + \delta w + f_2, \tag{2}$$

which was studied in [1]. In particular, system (2) with $\beta \neq 0$ and $\gamma \neq 0$ was reduced in [2] to the equation

$$\omega_{xy} + m\omega_x + n\omega_y + r\omega = f \tag{3}$$

for the functions $\omega = u$ and $\omega = v$, respectively. Note also that the problem on the relationship between the group properties of system (2) and Eq. (3) was considered in the monograph [3, pp. 185–191].

The aim of the present paper is to obtain sufficient conditions on the coefficients of system (1) under which there exists a third-order Bianchi equation

$$\theta_{xyz} + A\theta_{xy} + B\theta_{yz} + C\theta_{xz} + D\theta_x + E\theta_y + F\theta_z + G\theta = \Phi \tag{4}$$

satisfied by every function $\theta = u$ that is the first components of a solution (u, v, w) of system (1); we also obtain similar conditions for the second and third components v and w of the solutions.

1. In the derivation of Eq. (4) for the function $\theta = u$, we assume that $ab \neq 0$. We differentiate the first equation of system (1) with respect to z and express the derivative w_z in the resulting equation from the third equation of the system. Then we differentiate the equation thus obtained with respect to y and arrive at the equation

$$u_{xyz} = (a_z + bf)_y v + (a_z + bf)v_y + a_y v_z + av_{zy} + b_z w_y + b_{zy} w + (be)_y u + (be)u_y. \tag{5}$$

The right-hand side of Eq. (5) is a linear combination of the unknown functions v, v_y, v_z, v_{zy}, w_y, w, u, and u_y. Let us show that the first five of these functions can be expressed as linear combinations